Unitary Symmetry and Weak Interactions. II. SU(3) Transformation Properties*

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It is shown that if weak interactions are T-L invariant and satisfy the $\Delta T=\frac{1}{2}$ rule, their SU(3) transformation properties are severely limited. The nonleptonic Hamiltonian must belong to an octet, and the leptonic Hamiltonian is restricted to an octet and decuplet. To prove this result, the matrix elements of certain spin-type operators are expressed in terms of the Casimir operators for SU(3). T-L invariance and the $\Delta T=\frac{1}{2}$ rule impose constraints upon the eigenvalues of these Casimir operators and hence limit weak interactions in the manner described above. The converse of this result is not true: for example, a Hamiltonian belonging to an octet is not necessarily T-L invariant.

1. INTRODUCTION

In the first paper[1] of this series, a theory of weak interactions is developed from the viewpoint of unitary symmetry.[2] Leptonic and nonleptonic interactions are made to satisfy a symmetry principle known as T–L invariance, but nothing is assumed about their properties under general transformations of SU(3). Here we wish to show that T–L invariance has a strong influence upon this behavior. When combined with the $\Delta T=\frac{1}{2}$ rule, it forces the nonleptonic decay Hamiltonian to transform according to the eight-dimensional representation,[3] and restricts leptonic decays to the eight- and ten-dimensional ones. Notice, however, that the converse is not true: a Hamiltonian belonging to an octet is not necessarily T–L invariant.

The concept of T–L invariance arises from the fact that members of a unitary multiplet can be classified in three different ways.[4,5] One classification is based on the usual quantum numbers (T,T_3,Y_T) of isotopic spin and hypercharge, and the other two upon (K,K_3,Y_K) and (L,L_3,Y_L), respectively.[6] Any transformation which rearranges the members of a multiplet will lead to a corresponding transformation among these quantum numbers. Now, if it is to generate a symmetry principle for a particular class of phenomena, the transformation must always relate one charge conserving process to another; more precisely, it must leave invariant the appropriate condition for conservation of electric charge.

As an example, consider the charge symmetry operation

$$p\leftrightarrow n;\quad \Sigma^+\leftrightarrow\Sigma^-;\quad \pi^+\leftrightarrow\pi^-,$$

which is equivalent to

$$(T,T_3,Y_T)\rightarrow(T,\,-T_3,\,Y_T). \tag{1}$$

It can always be applied to strong interactions, but because it transforms $\Sigma^+\rightarrow n\pi^+$ into $\Sigma^-\rightarrow p\pi^-$, charge symmetry has no meaning for weak interactions. Another way of stating this result is to observe that the equation

$$\Delta Q\equiv\Delta T_3+\tfrac{1}{2}\Delta Y_T=0$$

is invariant under (1) only when ΔT_3 and ΔY_T are both zero.

To determine possible symmetry properties of weak interactions, we note that the above quantum numbers are not all independent of one another. By means of the relations,[6]

$$2T_3=Y_L-Y_K;\quad 2K_3=Y_T-Y_L;\quad 2L_3=Y_K-Y_T, \tag{2}$$

$$Y_T+Y_K+Y_L=0, \tag{3}$$

the third components of the spins can be expressed in terms of two types of hypercharge, for example, Y_T and Y_L. From the usual expression for electric charge, $Q=T_3+\frac{1}{2}Y_T$, it follows that

$$Y_K=-Q,$$

and hence that

$$\Delta Q=\Delta Y_T+\Delta Y_L. \tag{4}$$

Since ΔQ is not zero in leptonic decays, we are restricted to transformations which leave (4) invariant.

There is one very simple transformation which meets this requirement, namely,

$$Y_T\leftrightarrow Y_L. \tag{5a}$$

As a consequence of (2) and (3), it is equivalent to

$$T_3\leftrightarrow -L_3;\quad K_3\rightarrow -K_3;\quad Y_K\rightarrow Y_K. \tag{5b}$$

By augmenting (5a) and (5b) with an appropriate transformation for the total spins, we arrive at the definition of the T–L transformation[1]:

$$\begin{aligned}(T,T_3,Y_T)&\leftrightarrow(L,\,-L_3,\,Y_L),\\(K,K_3,Y_K)&\leftrightarrow(K,\,-K_3,\,Y_K).\end{aligned} \tag{6}$$

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[1] S. P. Rosen, Phys. Rev. Letters 12, 408 (1964).

[2] M. Gell-Mann, California Institute of Technology Report No. CTSL-20, 1961 (unpublished); Y. Ne'eman, Nucl. Phys. 26, 222 (1961).

[3] Octet transformation properties are taken as a basic assumption in several theories of weak interactions, for example: N. Cabibbo, Phys. Rev. Letters 10, 531 (1963); 12, 62 (1964); B. W. Lee, *ibid.* 12, 83 (1964); B. W. Lee and S. L. Glashow (to be published); M. Gell-Mann, Phys. Rev. Letters 12, 155 (1964); and S. Okubo (to be published).

[4] S. Meshkov, C. A. Levinson, and H. J. Lipkin, Phys. Rev. Letters 10, 361 (1963).

[5] S. P. Rosen, Phys. Rev. Letters 11, 100 (1963).

[6] S. P. Rosen, J. Math. Phys. 5, 289 (1964). We use the definition of L-spin given in this paper rather than the one in Ref. 5.

It is interesting to note that T–L invariance can be interpreted as a demand that weak interactions be symmetric with respect to two of the three classification schemes for a unitary multiplet.

Up to now our arguments have not been dependent on a specific representation of SU(3). Why then does T–L invariance restrict the behavior of weak interactions? The answer lies in the fact that relations between total spins, T, K, L, unlike those between T_3, K_3, L_3 [see (2) and (3)], vary from one multiplet to another. As will be shown below, the operators $\mathbf{L}^2$ and $\mathbf{K}^2$ can be expressed in terms of $\mathbf{T}^2$ and the two Casimir operators of SU(3). The eigenvalues of these Casimir operators are different in different representations, and so the values of L and K corresponding to a fixed value of T (e.g., $T=\frac{1}{2}$) will be different in different multiplets. Since T–L invariance places a constraint on L [see (6)], it will obviously limit the multiplets to which weak interactions may belong.

Identities relating the spin operators to the Casimir operators are derived in the next section, and they are then used to determine the transformation properties of weak interactions (see Sec. 3). Further consequences are discussed in the last section.

2. RELATIONS BETWEEN SPIN OPERATORS

In previous papers,[5–7] we have worked with the infinitesimal generators of U(3); here it will be convenient to work directly with the generators of SU(3). Denoting the former by $A_\nu{}^\mu$ and the latter by $B_\nu{}^\mu$, we have a simple relation between them:

$$B_\nu{}^\mu=A_\nu{}^\mu-\tfrac{1}{3}\delta_\nu{}^\mu M_1, \tag{7}$$

where[8]

$$M_1=A_1{}^1+A_2{}^2+A_3{}^3. \tag{8}$$

M_1 commutes with every $A_\nu{}^\mu$, and so $B_\nu{}^\mu$ satisfies the same commutation rules[8]

$$[B_\beta{}^\alpha,B_\nu{}^\mu]=\delta_\beta{}^\mu B_\nu{}^\alpha-\delta_\nu{}^\alpha B_\beta{}^\mu \tag{9}$$

and the same unitary restriction

$$(B_\nu{}^\mu)^\dagger=B_\mu{}^\nu \tag{10}$$

as $A_\nu{}^\mu$.

As a consequence of (9) and (10), the generators of SU(3) can be divided into three sets, each containing an angular momentum type operator and a corresponding hypercharge. They are[6]:

$$\begin{aligned}
&T_+=(T_-)^\dagger=-B_1{}^2,\quad T_3=\tfrac{1}{2}(B_2{}^2-B_1{}^1),\quad Y_T=B_3{}^3,\\
&K_+=(K_-)^\dagger=-B_2{}^3,\quad K_3=\tfrac{1}{2}(B_3{}^3-B_2{}^2),\quad Y_K=B_1{}^1,\\
&L_+=(L_-)^\dagger=-B_3{}^1,\quad L_3=\tfrac{1}{2}(B_1{}^1-B_3{}^3),\quad Y_L=B_2{}^2,
\end{aligned} \tag{11}$$

where

$$T_\pm=T_1\pm iT_2,\quad K_\pm=K_1\pm iK_2,\quad L_\pm=L_1\pm iL_2. \tag{12}$$

The first set of operators in (11) is identified with isotopic spin and the usual hypercharge

$$Y_T=B+S, \tag{13}$$

where B denotes baryon number and S strangeness. Equations (2) and (3) serve to identify the third components of spin and the hypercharges of the other two sets.

One relation between the total spins can be derived from the operator $\mathbf{V}$, which is defined as

$$\begin{aligned}
&V_+=(V_-)^\dagger=-A_3{}^2A_1{}^3,\\
&V_3=\tfrac{1}{2}(A_3{}^2A_2{}^3-A_3{}^1A_1{}^3)\equiv\tfrac{1}{2}(\mathbf{K}^2-\mathbf{L}^2+T_3+\tfrac{3}{2}T_3Y_T),\\
&V_\pm=V_1\pm iV_2.
\end{aligned} \tag{14}$$

The commutation relations between components of $\mathbf{T}$ and components of $\mathbf{V}$, e.g.,

$$[T_3,V_\pm]=\pm V_\pm,$$

imply that $\mathbf{V}$ is a vector operator in isotopic spin space.[9] It then follows that[9]

$$\begin{aligned}
\langle T,T_3,Y_T(\mu)|\mathbf{V}|T,T_3',Y_T'(\mu')\rangle
&=(1/T(T+1))\langle T,T_3,Y_T(\mu)|\mathbf{T}|T,T_3',Y_T(\mu)\rangle\\
&\quad\times\langle T,T_3',Y_T(\mu)|\mathbf{T}\cdot\mathbf{V}|T,T_3',Y_T'(\mu')\rangle,
\end{aligned} \tag{15}$$

where (μ) and (μ') indicates representations of SU(3). In particular, the component V_3 yields

$$\begin{aligned}
\langle T,T_3,Y_T(\mu)|\mathbf{K}^2-\mathbf{L}^2|T,T_3,Y_T(\mu)\rangle
&=(2T_3/T(T+1))\langle T,T_3,Y_T(\mu)|\mathbf{T}\cdot\mathbf{V}|T,T_3,Y_T(\mu)\rangle\\
&\quad-T_3(1+\tfrac{3}{2}Y_T).
\end{aligned} \tag{16}$$

To evaluate the expectation value of $\mathbf{T}\cdot\mathbf{V}$, we introduce the Casimir operators for SU(3):

$$\begin{aligned}
&\mathfrak{M}_2=B_\nu{}^\mu B_\mu{}^\nu,\\
&\mathfrak{M}_3=B_\nu{}^\mu B_\lambda{}^\nu B_\mu{}^\lambda-\tfrac{3}{2}B_\nu{}^\mu B_\mu{}^\nu.
\end{aligned} \tag{17}$$

Their eigenvalues in a representation $D(\mu_1,\mu_2)$ are[10]

$$\begin{aligned}
\mathfrak{M}_2(\mu_1,\mu_2)&=\tfrac{1}{3}[\mu_1{}^2+\mu_2{}^2+(\mu_1+\mu_2)^2+6(\mu_1+\mu_2)],\\
\mathfrak{M}_3(\mu_1,\mu_2)&=\tfrac{1}{9}(\mu_2-\mu_1)[(\mu_1+2\mu_2)(2\mu_1+\mu_2)\\
&\quad+9(\mu_1+\mu_2+1)].
\end{aligned} \tag{18}$$

It is tedious but not difficult to prove the identity[11]

$$\mathfrak{M}_3\equiv 6\mathbf{T}\cdot\mathbf{V}-3\mathbf{T}^2(\tfrac{3}{2}Y_T+1)+\tfrac{3}{4}Y_T(\mathfrak{M}_2+2)-\tfrac{3}{8}Y_T{}^3. \tag{19}$$

Combining (16) and (19), we obtain

$$\begin{aligned}
\langle T,T_3,Y_T(\mu)|\mathbf{K}^2-\mathbf{L}^2|T,T_3,Y_T(\mu)\rangle
&=(T_3/3T(T+1))\{\mathfrak{M}_3(\mu_1,\mu_2)\\
&\quad-\tfrac{3}{4}Y_T(\mathfrak{M}_2(\mu_1,\mu_2)+2)+\tfrac{3}{8}Y_T{}^3\}.
\end{aligned} \tag{20}$$

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[7] S. P. Rosen, Phys. Rev. **132**, 1234 (1963).

[8] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1961).

[9] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1959), pp. 59–61.

[10] The Casimir operators of SU(3) can be expressed in terms of the corresponding operators for U(3) (see S. Okubo, Ref. 8) by means of Eq. (7). Their eigenvalues in Eq. (18) can then be calculated by taking $\mu_1=f_1-f_2$, $\mu_2=f_2-f_3$; because $f_1\geqslant f_2\geqslant f_3$ (see Ref. 8), it follows that μ_1 and μ_2 are always non-negative.

[11] G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963), prove a similar identity. Our use of the operator $\mathbf{V}$ to prove Eq. (20) is modeled on Sec. (1.D) of this paper.

In a similar fashion we can prove two other identities:

$$\langle L,L_3,Y_L(\mu)|\mathbf{T}^2-\mathbf{K}^2|L,L_3,Y_L(\mu)\rangle = (L_3/3L(L+1))\{\mathfrak{M}_3(\mu_1,\mu_2) -\tfrac{3}{4}Y_L(\mathfrak{M}_2(\mu_1,\mu_2)+2)+\tfrac{3}{8}Y_L{}^3\}, \quad (21)$$

and

$$\langle K,K_3,Y_K(\mu)|\mathbf{L}^2-\mathbf{T}^2|K,K_3,Y_K(\mu)\rangle = (K_3/3K(K+1))\{\mathfrak{M}_3(\mu_1,\mu_2) -\tfrac{3}{4}Y_K(\mathfrak{M}_2(\mu_1,\mu_2)+2)+\tfrac{3}{8}Y_K{}^3\}. \quad (22)$$

The second of these will be very useful in the discussion of nonleptonic decays.

Since $\mathbf{T}^2$, $\mathbf{K}^2$, $\mathbf{L}^2$ do not commute with one another, the eigenstates of $\mathbf{K}^2$ and $\mathbf{L}^2$ will, in general, be linear combinations of isotopic spin eigenstates. Thus the states $|T,T_3,Y_T(\mu)\rangle$, $|K,K_3,Y_K(\mu)\rangle$, $|L,L_3,Y_L(\mu)\rangle$ cannot always refer to one and the same member of a unitary multiplet.[5–7] There are, of course, exceptions to this, and when they occur, the member in question must be a simultaneous eigenstate of all three spin operators.

To prove the last statement, we use the identity[6]

$$\mathbf{T}^2+\mathbf{K}^2+\mathbf{L}^2\equiv\tfrac{1}{2}\mathfrak{M}_2+\tfrac{1}{4}(Y_T{}^2+Y_K{}^2+Y_L{}^2). \quad (23)$$

It implies that if a member of a given unitary multiplet is an eigenstate of two spin operators, it must also be an eigenstate of the third. Thus the member is either an eigenstate of isotopic spin alone, or it is an eigenstate of all three spins. We shall find that, as a consequence of T–L invariance, weak interactions fall into the latter category.

3. TRANSFORMATION PROPERTIES OF WEAK INTERACTIONS

From a phenomenological point of view, weak interactions may transform according to several different representations of SU(3), and the most general Hamiltonian will be a sum of terms $\mathfrak{H}(\mu)$, each belonging to a specific unitary multiplet:

$$\mathfrak{H}=\sum_\mu \mathfrak{H}(\mu). \quad (24)$$

Since the T–L transformation is a member of the SU(3) group,[12] it cannot relate $\mathfrak{H}(\mu)$ to $\mathfrak{H}(\mu')$ when (μ) and (μ') are two different multiplets; however, it can determine whether a particular $\mathfrak{H}(\mu)$ does, or does not, appear in the summation of (24). We therefore ask the following question: given that the $\Delta T=\frac{1}{2}$ rule and T–L invariance are obeyed in weak interactions, what representations of SU(3) can appear in the effective Hamiltonian?

The nonleptonic interaction Hamiltonian can be written as the sum of two parts $\mathfrak{H}_\pm$ where one is the Hermitian conjugate of the other. From their quantum numbers,

$$T_3=\mp\tfrac{1}{2},\quad Y_T=\pm1;\quad K_3=\pm1,\quad Y_K=0;\quad L_3=\mp\tfrac{1}{2},\quad Y_L=\mp1, \quad (25)$$

it can be seen that $\mathfrak{H}_+$ and $\mathfrak{H}_-$ are interchanged by the T–L transformation. The $\Delta T=\frac{1}{2}$ rule and T–L invariance imply that

$$T=L=\tfrac{1}{2}, \quad (26)$$

and hence that each $\mathfrak{H}(\mu)$ in the decomposition of (24) will be an eigenstate of K spin (see the discussion at the end of Sec. 2). For a given representation, the eigenvalue K can be determined in two distinct ways, either from the identity in (23) or from Eqs. (20)–(22). Since both methods must yield the same value for K, we obtain equations of constraint upon the characteristic numbers (μ_1,μ_2).

Substituting (25) and (26) into (22), we find

$$\mathfrak{M}_3(\mu_1,\mu_2)=0. \quad (27)$$

Since μ_1 and μ_2 are non-negative,[10,13] it follows from (18) that the only solution of (27) is

$$\mu_1=\mu_2. \quad (28)$$

We now use (23) to eliminate $\mathbf{K}^2$ from Eq. (20), and, with the aid of (25)–(28), we obtain an equation for μ_1:

$$\mathfrak{M}_2(\mu_1,\mu_1)=6. \quad (29)$$

The only acceptable solution is

$$\mu_1=1,$$

and hence the nonleptonic Hamiltonian can transform only as a member of the representation $D(1,1)$, i.e., the eightfold representation.[13]

Leptons are assumed to be unitary singlets, and the behavior of leptonic decays will therefore be determined by the weak currents of strongly interacting particles.[14] Since the strangeness conserving current $J^{(0)}$ and the strangeness violating current $J^{(1)}$ have quantum numbers

$$T_3=1,\quad Y_T=0;\quad K_3=-\tfrac{1}{2},\quad Y_K=-1;\quad L_3=-\tfrac{1}{2},\quad Y_L=1, \quad (30a)$$

and

$$T_3=\tfrac{1}{2},\quad Y_T=1;\quad K_3=\tfrac{1}{2},\quad Y_K=-1;\quad L_3=-1,\quad Y_L=0, \quad (31a)$$

respectively, the T–L transformation interchanges them. If $J^{(0)}$ obeys the $\Delta T=1$ rule and $J^{(1)}$ the $\Delta T=\frac{1}{2}$

[12] The T–L transformation is related to a more general transformation discussed by B. d'Espagnat and J. Prentki, Nuovo Cimento **24**, 497 (1962), and to the Weyl reflections of A. J. Macfarlane, E. C. G. Sudarshan and C. Dullemond, Nuovo Cimento **30**, 845 (1964). It will be examined in detail in another paper of this series.

[13] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).

[14] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

rule, then if follows from T–L invariance that

$$T=1,\quad L=\tfrac{1}{2},\quad \text{for}\quad J^{(0)}, \tag{30b}$$

and

$$T=\tfrac{1}{2},\quad L=1,\quad \text{for}\quad J^{(1)}. \tag{31b}$$

Again we observe that each $\mathfrak{H}(\mu)$ in the appropriate decomposition of the currents is an eigenstate of K spin, and that the two methods for calculating K will place a constraint upon (μ_1,μ_2).

Now consider $J^{(0)}$: from Eqs. (30) and (23) we find

$$K(K+1)=\tfrac{1}{2}\mathfrak{M}_2(\mu_1,\mu_2)-\tfrac{9}{4}, \tag{32}$$

and from (21)

$$K(K+1)=\tfrac{1}{6}\mathfrak{M}_3(\mu_1,\mu_2)+\tfrac{3}{4}. \tag{33}$$

If (32) and (33) are to yield the same value of K, then

$$\mathfrak{M}_3(\mu_1,\mu_2)-3\mathfrak{M}_2(\mu_1,\mu_2)+18=0. \tag{34}$$

This equation has three solutions,

$$\begin{aligned}&\text{(i)}\ \mu_1+2\mu_2+9=0,\\ &\text{(ii)}\ \mu_1-\mu_2+6=0,\\ &\text{(iii)}\ 2\mu_1+\mu_2-3=0,\end{aligned} \tag{35}$$

the first of which cannot be satisfied because μ_1 and μ_2 are never negative. From (18) and (32), the second solution leads to

$$K=\mu_1+\tfrac{7}{2}. \tag{36}$$

Now in any representation $D(\mu_1,\mu_2)$, the maximum value of K is[15] $\tfrac{1}{2}(\mu_1+\mu_2)$; for representations satisfying solution (ii), this becomes

$$K_{\max}=\mu_1+3. \tag{37}$$

Since the value of K in (36) is incompatible with (37), we must reject the second solution of (35). On the other hand, the third solution gives a value of K, namely,

$$K=\tfrac{3}{2}-\mu_1, \tag{38}$$

which is compatible with the corresponding maximum $\tfrac{1}{2}(3-\mu_1)$. Therefore $J^{(0)}$ can transform according to any representation which satisfies

$$2\mu_1+\mu_2=3. \tag{39}$$

The same conclusion holds for $J^{(1)}$.

[15] See Ref. 11. The parameters used there are related to ours by $p=\mu_1+\mu_2$, $q=\mu_2$.

Because μ_1 and μ_2 are non-negative integers, there are only two acceptable solutions of (39). They are

$$\begin{aligned}&\mu_1=\mu_2=1,\\ &\mu_1=0,\quad \mu_2=3,\end{aligned} \tag{40}$$

and correspond to the eightfold and tenfold (10*) representations,[13] respectively. In general, the strangeness conserving current will be a linear combination of these two representations, and because the T–L transformation belongs to SU(3),[12] the strangeness violating current will be exactly the same linear combination.

4. CONCLUSIONS

We have now shown that the $\Delta T=\tfrac{1}{2}$ rule and T–L invariance force nonleptonic weak interactions to transform as a member of a unitary octet, and restrict the leptonic ones to an octet and decuplet. There still remains the question of the converse.

If the nonleptonic Hamiltonian were a member of an octet, it would, in general, transform as a linear combination of K_1 and K_2.[16] Now, under the T–L transformation[1]

$$K^0\rightarrow-\bar{K}^0,$$

and so the only T–L invariant combination is $K_2\equiv(1/\sqrt{2})(K^0-\bar{K}^0)$. Therefore the Hamiltonian will only be T–L invariant when the K_1 component vanishes.

Similarly, if the leptonic Hamiltonian were a member of an octet, it would transform as a linear combination of Σ^+ and p (leptons are taken to be unitary singlets). Again there is only one T–L invariant combination, namely,[1] (Σ^+-p) and so the Hamiltonian need not be T–L invariant. The same type of argument applies to the decuplet.

Thus we see that the converse of our result is not true: the assumption that weak interactions transform according to the eightfold representation of SU(3) does not imply that they are T–L invariant.

One final point: it is not difficult to show that the T–L transformation transforms a $\Delta S=2$ current into a $\Delta S=-\Delta Q$ current. Therefore, if weak interactions are T–L invariant, these types of transition will either both occur or both not occur: one cannot occur without the other.[17]

[16] See, for example, the paper of M. Gell-Mann in Ref. 3.
[17] See the paper of d'Espagnat and Prentki cited in Ref. 12.